by the following process:

$$a = a_1 b - r_1, \qquad b = a_2 r_1 - r_2, \qquad r_1 = a_3 r_2 - r_3, \text{ etc.}$$

Here a_1, a_2, $\cdots$ are positive integers, and, if we suppose a, b to have been taken positive, $b > r_1 > r_2 > \cdots > 0$. Show that if a/b be but slightly larger than a positive integer or zero, the numerators (as likewise the denominators) of the successive convergents will for a time be in arithmetical progression, and determine how long this phenomenon will continue.

2696. Proposed by L. E. LUNN, Heron Lake, Minnesota.

An air pipe 18 inches in diameter passes diagonally through a room from one lower corner to the opposite upper corner leaving through elliptical openings in the floor and ceiling, so that the ellipses are tangent to two boundaries of the floor and to the two opposite boundaries of the ceiling. If the room is 10 x 12 x 8, find the remaining cubic capacity of the room.

2697. Proposed by H. S. UHLER, Yale University.

Show how to reduce the left-hand members of the following identities to their respective right members:

$$\sin^2 (x + \tfrac{1}{2}y) - \sin (x + \tfrac{3}{2}y) \sin (x - \tfrac{1}{2}y) = \sin^2 y,$$

$$\sin (x + y) \sin (x + \tfrac{1}{2}y) - \sin x \sin (x + \tfrac{3}{2}y) = \sin \tfrac{1}{2}y \sin y,$$

$$\sin x \sin (x + \tfrac{1}{2}y) - \sin (x - \tfrac{1}{2}y) \sin (x + y) = \sin \tfrac{1}{2}y \sin y.$$

2698. Proposed by WARREN WEAVER, Throop College of Technology, Pasadena, California.

An urn contains N balls numbered from 1 to N. Of these n are drawn out and are arranged linearly according to the numbers on each. A certain ball is observed to be the kth in this line. What is the most probable number written on this ball?

SOLUTIONS OF PROBLEMS.

490 (Algebra). Proposed by HENRY HEATON, Atlantic, Iowa.

Show that $\sin 3° = \frac{1}{16}(\sqrt{30} + \sqrt{10} - \sqrt{6} - \sqrt{2}) + \frac{1}{8}(\sqrt{5 + \sqrt{5}} - \sqrt{15 + 3\sqrt{5}})$.

SOLUTION BY S. E. RASOR, The Ohio State University.

Since $3° = 12° - 9°$, $12° = 30° - 18°$, and, for $\theta = 18°$, $2\theta = 90° - 3\theta$, the sine and the cosine of 9°, 12°, 18°, and thus sin 3° may be found as follows:

We have

$$\sin 2\theta = 2 \sin \theta \cos \theta = \sin (90° - 3\theta) = \cos 3\theta = 4 \cos^3 \theta - 3 \cos \theta,$$

$$4 \sin^2 \theta + 2 \sin \theta = 1,$$

and

$$\sin 18° = \tfrac{1}{4}(\sqrt{5} - 1), \quad \cos 18° = \tfrac{1}{4}\sqrt{10 + 2\sqrt{5}}.$$

Also from the identities, $\sin \frac{1}{2}A \pm \cos \frac{1}{2}A = \pm \sqrt{1 \pm \sin A}$ for $A = 18°$, we have

$$\sin 9° = \tfrac{1}{4}(\sqrt{3 + \sqrt{5}} - \sqrt{5 - \sqrt{5}}), \qquad \cos 9° = \tfrac{1}{4}(\sqrt{3 + \sqrt{5}} + \sqrt{5 - \sqrt{5}}).$$

Also,

$$\sin 12° = \sin (30° - 18°) = \sin 30° \cos 18° - \cos 30° \sin 18° = \tfrac{1}{8}(\sqrt{10 + 2\sqrt{5}} - \sqrt{15} + \sqrt{3}),$$

$$\cos 12° = \tfrac{1}{8}(\sqrt{30 + 6\sqrt{5}} + \sqrt{5} - 1).$$

Therefore,

$$\sin 3° = \sin (12° - 9°) = \sin 12° \cos 9° - \cos 12° \sin 9°$$

$$= \left(\frac{\sqrt{10 + 2\sqrt{5}} - \sqrt{15} + \sqrt{3}}{8}\right)\left(\frac{\sqrt{3 + \sqrt{5}} + \sqrt{5 - \sqrt{5}}}{4}\right) - \left(\frac{\sqrt{30 + 6\sqrt{5}} + \sqrt{5} - 1}{8}\right)\left(\frac{\sqrt{3 + \sqrt{5}} - \sqrt{5 - \sqrt{5}}}{4}\right)$$

$$= \tfrac{1}{16}(\sqrt{30} + \sqrt{10} - \sqrt{6} - \sqrt{2}) + \tfrac{1}{16}(\sqrt{10 + 4\sqrt{5}} + \sqrt{10 - 4\sqrt{5}} - \sqrt{30 + 12\sqrt{5}} - \sqrt{30 - 12\sqrt{5}})$$

$$= \tfrac{1}{16}(\sqrt{30} + \sqrt{10} - \sqrt{6} - \sqrt{2}) + \tfrac{1}{16}(1 - \sqrt{3})(\sqrt{10 + 4\sqrt{5}} + \sqrt{10 - 4\sqrt{5}}).$$

By simplifying the expression in the last parenthesis and then collecting under the radical sign, we have, after a simple reduction,

$$\sin 3° = \tfrac{1}{16}(\sqrt{30} + \sqrt{10} - \sqrt{6} - \sqrt{2}) + \tfrac{1}{8}(\sqrt{5 + \sqrt{5}} - \sqrt{15 + 3\sqrt{5}}).$$

Also solved by L. E. Mensenkamp, J. L. Riley, Horace Olson, H. S. Uhler, H. C. Feemster, Florence Rae, R. M. Mathews, H. L. Agard, Gilbert A. Ault, Frank Irwin, C. E. Githens, A. M. Harding, and the Proposer.

491 (Algebra). Proposed by J. W. Lasley, University of North Carolina.

Solve the equations $xy = x^2 - y^2$ and $x^2 + y^2 = x^3 - y^3$ for x and y.

Solution by J. W. Baldwin, Ann Arbor, Michigan.

Solving $xy = x^2 - y^2$ for x in terms of y we have $x = \frac{1}{2}(1 + \sqrt{5})y$ and $x = \frac{1}{2}(1 - \sqrt{5})y$. These values substituted in $x^2 + y^2 = x^3 - y^3$ give, after simplification, $y^2(y - \frac{1}{2}\sqrt{5}) = 0$ (1) and $y^2(y + \frac{1}{2}\sqrt{5}) = 0$ (2). From (1), $y = 0, 0, \frac{1}{2}\sqrt{5}$ and from (2) $y = 0, 0, -\frac{1}{2}\sqrt{5}$. Hence, the corresponding values of x are $0, 0, (5 + \sqrt{5})/4$ and $0, 0, (5 - \sqrt{5})/4$. From (1) or (2) it is seen that two branches of the curve represented by the second equation pass through the origin. It is readily determined that these branches are imaginary and, hence, the origin is a conjugate point.

Also solved by S. E. Rasor, F. H. Holestin, Elgin E. Groseclose, H. N. Carleton, A. M. Harding, J. L. Riley, Polycarp Hansen, E. B. Escott, G. Y. Sosnow, J. Q. McNatt, O. S. Adams, Horace Olson, T. C. Amick, L. E. Lunn, Pauline Sperry and the Proposer.

433 (Calculus). Proposed by Louis O'Shaughnessy, University of Pennsylvania.

Solve the differential equation

$$\frac{d^{\frac{1}{2}}y}{dx^{\frac{1}{2}}} - \frac{y}{x} = 0.$$

I. Solution by Emil L. Post, New York City.

We have

$$x\frac{d^{\frac{1}{2}}y}{dx^{\frac{1}{2}}} = y. \tag{1}$$

Operating on both sides by $d^{\frac{1}{2}}/dx^{\frac{1}{2}}$ (see next problem), we have

$$\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\left[x\frac{d^{\frac{1}{2}}y}{dx^{\frac{1}{2}}}\right] = \frac{d^{\frac{1}{2}}y}{dx^{\frac{1}{2}}}. \tag{2}$$

But for any operation $f(D)$

$$f(D)u\cdot v = uf(D)v + \frac{du}{dx}\frac{f'(D)v}{1!} + \frac{d^2u}{dx^2}\frac{f''(D)v}{2!} + \cdots.$$

Let

$$f(D) = D^{\frac{1}{2}} = \frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}; \qquad u = x; \qquad v = \frac{d^{\frac{1}{2}}y}{dx^{\frac{1}{2}}}.$$

Then

$$\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\left[x\frac{d^{\frac{1}{2}}y}{dx^{\frac{1}{2}}}\right] = x\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\frac{d^{\frac{1}{2}}y}{dx^{\frac{1}{2}}} + \frac{1}{2}\frac{d^{-\frac{1}{2}}}{dx^{-\frac{1}{2}}}\frac{d^{\frac{1}{2}}y}{dx^{\frac{1}{2}}} = x\frac{dy}{dx} + \tfrac{1}{2}y. \tag{3}$$